

Condensation of fractional excitons, non-Abelian states in double-layer quantum Hall systems and Z_4 parafermions

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In this paper, we study a method to obtain non-Abelian FQH state through double-layer FQH states and fractional exciton condensation. In particular, we find that starting with the (330) double-layer state and then increasing the interlayer tunneling strength, we may obtain a single-layer non-Abelian FQH state $S(330)$. We show that the $S(330)$ state is actually the Z_4 parafermion Read-Rezayi state. We also calculate the edge excitation of the $S(330)$ state.

I. INTRODUCTION

Non-Abelian fractional quantum Hall (FQH) states are a new class of FQH states whose excitations carry non-Abelian statistics.¹⁻³ The internal order of non-Abelian states is so different from the symmetry breaking orders that totally new approaches are needed to study them. Since the very beginning, two very effective yet very different approaches were introduced; one is based on conformal field theory¹ and the other is based on projective construction.^{2,4} Both approaches allow us to calculate non-Abelian quasiparticles and edge states.^{1,3-7} The projective construction also allows us to calculate the effective bulk topological field theory for the non-Abelian FQH states. This class of quantum states is proposed to be a medium for fault tolerant quantum computation.⁸⁻¹⁰

However, it is non trivial to realize non-Abelian states; in particular, most observed FQH states are believed to be Abelian FQH states. But by utilizing the special form of electron-electron interaction in the second Landau level, a non-Abelian FQH state, the Moore-Read Pfaffian state,^{1,11,12} may be realized as the $\nu = 5/2$ FQH state.¹³ The $\nu = 12/5$ state may also be a non-Abelian state^{14,15}, although more experimental studies are needed to be sure.

In this paper, we will study another possible mechanism for realizing non-Abelian states, namely the possibility to make them in double layer systems. Experimentally, the hierarchal FQH states observed in double systems are quite different from those in the first Landau level. So there may be non-Abelian FQH states in double-layer systems.

In general, double-layer FQH states contain a class of fractional exciton excitations. A fractional exciton (f-exciton) is formed by a pair of fractionally charged quasiparticle and quasihole in each layer. Such a fractional exciton may carry fractional statistics. If we start with an Abelian double-layer state, the condensation of fractional excitons may generate a non-Abelian state.

The condensation of fractional excitons and the resulting non-Abelian Moore-Read Pfaffian state has been studied for the 331 double-layer state.¹⁶ Such a phenomenon is closely related to the transition between the weak-pairing p -wave to strong pairing p -wave BCS superconductors.¹⁷ In this paper, we show that the fractional exciton condensation in the 330 double-layer state generate the $k = 4$ Read-Rezayi parafermion state.

II. F-EXCITONS AND THEIR CONDENSATION

Let us consider a double-layer quantum Hall system of spin polarized electrons at filling fraction $2/n$, where n is odd. We will consider only the $T = 0$ quantum ground states. First we assume the interlayer tunneling and interlayer interaction to be zero. In this case the electrons in the two layers form two independent $\nu = 1/n$ Laughlin states, which is denoted as $(nn0)$ state. Such a state is a special case of more general (mnl) double-layer states whose wave function is given by

$$\Phi_{mnl}(\{z_i\}, \{w_i\}) \propto \prod_{i < j} (z_i - z_j)^m (w_i - w_j)^n \prod_{i,j} (z_i - w_j)^l e^{-\frac{1}{4} \sum_i (|z_i|^2 + |w_i|^2)}, \quad (1)$$

where the complex number z_i (w_i) are electron coordinates in the first (second) layer.

In this paper, we will also consider the situation where n is even. In this case, electrons must carry Bose statistics. So the bosonic “electrons” will form two independent $\nu = 1/n$ Laughlin states in the two layers for even n . The wave function still has the form in eqn. (1).

Now let us still assume the interlayer tunneling to be zero, but increase the repulsive interaction between electrons in the two different layers. Clearly, when the interlayer repulsive interaction is very strong, the electrons will all stay in one of the two layers and from a single-layer $\nu = 2/n$ state. The single-layer $\nu = 2/n$ state is

a charge imbalanced state since the electron densities in the two layers are different. The charge imbalanced state spontaneously breaks the symmetry of exchanging the two layers. Thus the state for small interlayer repulsion and the state for large interlayer repulsion have different symmetries and belong to different phases. This suggests that as we increase interlayer repulsion from zero, we induce a quantum phase transition at a critical interlayer repulsion. What is this quantum phase transition?

To understand the phase transition (or the instability of the $(nn0)$ state) induced by interlayer repulsion, let us consider the following excitation in the $(nn0)$ state which is formed by a charge e/n quasiparticle in one layer and a charge $-e/n$ quasihole in the other layer. We will call such an excitation a fractionalized exciton (f-exciton). Clearly, the presence of a finite density of f-excitons will cause a charge imbalance between the two layers. Thus a strong interlayer repulsion will generate many f-excitons and cause a charge imbalanced state such as the single-layer $\nu = 2/n$ state.

This consideration suggests that if we increase interlayer repulsion in the $(nn0)$ state, the energy gap for the f-excitons will be reduced. When the energy gap of the f-exciton is reduced to zero, the $(nn0)$ state will become unstable and a quantum phase transition will happen. Such a quantum phase transition can be studied using the approach developed in Ref. 16. The effective theory for the transition is found to be a Ginzburg-Landau-Chern-Simons (GLCS) theory

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - m^2|\phi|^2 - g|\phi|^4 - \frac{n}{2} \frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}. \quad (2)$$

Such a low energy effective theory describes the critical point at the transition.

We would like to make a few remarks:

- (a) Near the phase transition, both f-excitons and anti-f-excitons become gapless. This is why the effective GLCS theory is relativistic. The field ϕ describes both f-excitons and anti-f-excitons.
- (b) The f-excitons obey fractional statistics with $\theta = 2\pi/n$. The $U(1)$ gauge field a_μ with the Chern-Simons term is introduced to reproduce the fractional statistics.
- (c) The energy gap of the f-excitons is m .
- (d) As we increase the interlayer repulsion from zero, m^2 will decrease. For large interlayer repulsion, m^2 will be reduced to a negative value, and cause a phase transition from the $\phi = 0$ phase to $\phi \neq 0$ phase.
- (e) We note that in the $\phi \neq 0$ phase the densities of f-excitons and anti-f-excitons are equal. So both the $\phi = 0$ phase and $\phi \neq 0$ phases have an equal electron density in the two layers and are charge balanced states. The charge imbalanced state mentioned above will appear for even stronger interlayer repulsion.

In the presence of inter-layer repulsion, the dispersion of the f-excitons may have several different possible

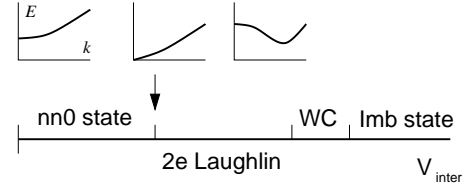


FIG. 1: Possible dispersions of low lying charge neutral excitations. In the $nn0$ state, the low lying charge neutral excitations are the f-excitons. Such dispersion for the f-excitons leads to the transition between the $nn0$ state and the charge $2e$ Laughlin state. At larger V_{inter} , the Wigner crystal (WC) and a charge imbalanced state (lmb state) may appear.

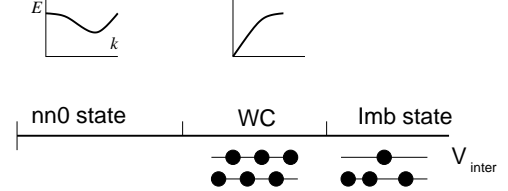


FIG. 2: Another set of possible dispersions of low lying charge neutral excitations. In the $nn0$ state, the low lying charge neutral excitations are the f-excitons. Such dispersion for the f-excitons leads to the transition between the $nn0$ state and the WC state.

forms. One possible dispersion relation is illustrated in Fig. 1. The effective theory (2) applies to such a situation.

Another possible dispersion relation for the f-excitons is illustrated in Fig. 2. In this case, the effective theory (2) does not apply. Such case will likely lead to the WC state.

III. DOUBLE-LAYER QUANTUM HALL STATES AND THEIR PHASE TRANSITIONS

Let us assume that the situation described by Fig. 1 is realized and the effective theory (2) is valid. Clearly, the $\phi = 0$ phase is the $(nn0)$ state. What is the $\phi \neq 0$ state? In Ref. 16, it is shown that the $\phi \neq 0$ state is the Laughlin state for charge $2e$ electron pairs. The effective filling fraction for the $2e$ pairs is $\nu_{\text{eff}} = 1/2n$. The $\phi = 0$ to $\phi \neq 0$ transition at $m^2 = 0$ is believed to be a continuous quantum phase transition.^{18,19} Such a transition is a transition between the $(nn0)$ double layer state and the charge- $2e$ Laughlin state, which exists in the absence of interlayer tunneling.

In the presence of a finite interlayer tunneling, the electron numbers in each layer are no longer conserved separately. The effective theory (2) will contain extra terms to reflect this reduced symmetry. Note that an f-exciton is created by the operator $\phi \hat{M}$, where \hat{M} is an operator that creates $2/n$ units of a_μ flux. Thus $\phi^n \hat{M}^n$ creates n f-excitons, which correspond to an electron in one layer and a hole in the other layer. So the electron tunnel-

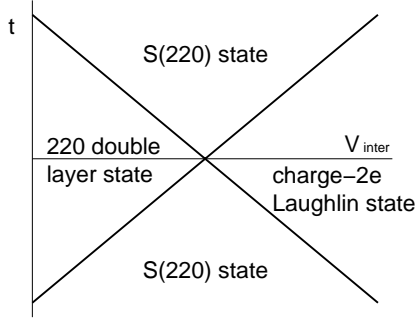


FIG. 3: The phase diagram for the (220) state, the charge-2e Laughlin state, and the $\mathcal{S}(220)$ state. The $\mathcal{S}(220)$ state is the Pfaffian state. t is the amplitude of the interlayer electron tunneling and V_{inter} is the strength of interlayer repulsion.

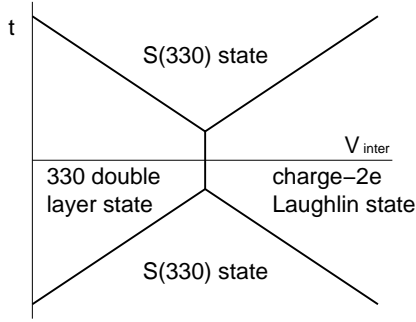


FIG. 4: The proposed phase diagram for the (330) state, the charge-2e Laughlin state, and $\mathcal{S}(330)$ state.

ing operator correspond to $\phi^n \hat{M}^n + h.c.$. Thus, with interlayer electron tunneling, the effective theory near the transition becomes

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - m^2|\phi|^2 - g|\phi|^4 - t(\phi^n \hat{M}^n + h.c.) - \frac{n}{2} \frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \quad (3)$$

where t is the amplitude of the interlayer electron tunneling.

When $n = 2$, the f-excitons happen to have Fermi statistics. So the effective theory for the (220) state can be mapped to a fermion model without the Chern-Simons term. The fermionic effective theory is exactly soluble which allows us to calculate the phase diagram for the (220) state (see Fig. 3).^{16,17} For the critical point at $t = m^2 = 0$, the gapless excitations are all neutral and are described by free massless Dirac fermions. For the critical point at $t \neq 0$, the neutral gapless excitations are described by free massless Majorana fermions.

When $n > 2$, the effective theory for the $(nn0)$ state cannot be solved. So we do not know the phase diagram, except that when $t = 0$ we believe that there is a continuous phase transition at $m^2 = 0$. It is possible that such a continuous phase transition is stable against a small interlayer tunneling. However, a large interlayer tunneling may induce a phase transition from the $(nn0)$ state to a

new state which will be called the single-layer $(nn0)$ state and denoted as $\mathcal{S}(nn0)$ state. This leads to the proposed phase diagram for the $(nn0)$ state (see Fig. 4).

IV. THE PROPERTIES OF THE $\mathcal{S}(nn0)$ STATE

What are the properties of the $\mathcal{S}(nn0)$ state? For large interlayer tunneling t , the electron state created by $\psi_{e1}^\dagger + \psi_{e2}^\dagger$ has a lower energy and the state created by $\psi_{e1}^\dagger - \psi_{e2}^\dagger$ has a higher energy. (Here ψ_{e1} and ψ_{e2} are the electron operators in the two layers.) So we expect that for large t , the $(nn0)$ state changes into a single-layer state where electrons are always in the even state $\psi_{e1} + \psi_{e2}$. This consideration allows us to guess the groundstate wave function of the $\mathcal{S}(nn0)$ state (which is induced by large interlayer tunneling from the $(nn0)$ state). Then from the guessed groundstate wave function, we can obtain the physical properties of the $\mathcal{S}(nn0)$ state.

We first note that the wave function of the $(nn0)$ state can be expressed as

$$\Phi_{nn0}(\{z_i\}, \{w_i\}) = \langle 0 | \prod_i \psi_{e1}(z_i) \psi_{e2}(w_i) | nn0 \rangle \quad (4)$$

where $|0\rangle$ is the state with no electron and the state $|nn0\rangle$ is the $(nn0)$ state. An electron can be in a mixed even $\psi_{e1} + \psi_{e2}$ and odd state $\psi_{e1} - \psi_{e2}$. After the electrons are projected into the even state $\psi_{e1} + \psi_{e2}$, the $(nn0)$ state changes into the single-layer $\mathcal{S}(nn0)$ state. The wave function for the $\mathcal{S}(nn0)$ state is then given by

$$\Phi_{\mathcal{S}(nn0)}(\{z_i\}) = \langle 0 | \prod_i [\psi_{e1}(z_i) + \psi_{e2}(z_i)] | nn0 \rangle \quad (5)$$

For even n , we have

$$\Phi_{\mathcal{S}(nn0)}(\{z_i\}) = \mathcal{S} \prod_{i,j} (z_{2i} - z_{2j})^n (z_{2i+1} - z_{2j+1})^n e^{-\frac{1}{4} \sum_i |z_i|^2} \quad (6)$$

and for odd n

$$\Phi_{\mathcal{S}(nn0)}(\{z_i\}) = \mathcal{A} \prod_{i,j} (z_{2i} - z_{2j})^n (z_{2i+1} - z_{2j+1})^n e^{-\frac{1}{4} \sum_i |z_i|^2}. \quad (7)$$

Here \mathcal{S} is the symmetrization operator and \mathcal{A} the anti-symmetrization operator. Note that $\prod_{i,j} (z_{2i} - z_{2j})^n (z_{2i+1} - z_{2j+1})^n e^{-\frac{1}{4} \sum_i |z_i|^2}$ is the wave function for the $(nn0)$ state with z_{2i} being the electron coordinates in the first layer and z_{2i+1} the coordinates in the second layer. So the symmetrization \mathcal{S} or the anti-symmetrization \mathcal{A} perform the projection into even states $\psi_{e1} + \psi_{e2}$.

Once we find an expression of the ground state wave function in terms of correlation, such as eqn. (5), then we can use the parton construction developed in Ref. 4 to find the bulk effective theory and the edge excitations. In other words, we can determine the topological order

from the expression of the ground state wave function (5).

In this paper, we will concentrate only on the edge states. What is the spectrum of the edge excitations of the $\mathcal{S}(nn0)$ state? Let M_0 be the total angular momentum of the ground state wave function $\Phi_{\mathcal{S}(nn0)}$. The number of low-energy edge states with a fixed total number of electrons and at angular momentum $M_0 + l$ is given by D_l . In appendix A, we present a calculation of D_l for the $\mathcal{S}(220)$ and $\mathcal{S}(330)$ state.

We find that the edge spectrum for the $\mathcal{S}(220)$ state (with an even number of electrons) is given by

$$D_l : 1, 1, 3, 5, 10, 16, 28, 43, 70$$

The electron operators has the following correlation

$$\langle \psi_e(t, 0) \psi_e^\dagger(0, 0) \rangle \sim \frac{1}{t^{g_e}}, \quad g_e = 2.$$

We find that the minimal charged quasiparticle ψ_q in the $\mathcal{S}(220)$ state carries a charge $e/2$. The quasiparticle operator has the following correlation

$$\langle \psi_q(t, 0) \psi_q(0, 0) \rangle \sim \frac{1}{t^{g_q}}, \quad g_q = 3/8.$$

The wavefunction of the $\mathcal{S}(220)$ state is just the the $\nu = 1$ Pfaffian state.¹ The edge theory for the $\nu = 1/2$ Pfaffian state was studied numerically in Ref. 12 which has an identical spectrum as the $\mathcal{S}(220)$ state discussed above.

The edge spectrum for the $\mathcal{S}(330)$ state (with an $4 \times$ integer number of electrons) is given by

$$D_l : 1, 1, 3, 6, 13, 23, 44, 75, 131, 215, 354, 561... \quad (8)$$

The electron operators have a correlation

$$\langle \psi_e(t, 0) \psi_e^\dagger(0, 0) \rangle \sim \frac{1}{t^{g_e}}, \quad g_e = 3$$

The minimally charged quasiparticle operator ψ_q carries a charge $e/6$ with a correlation

$$\langle \psi_q(t, 0) \psi_q^\dagger(0, 0) \rangle \sim \frac{1}{t^{g_q}}, \quad g_q = 1/2.$$

It turns out that the wavefunction of the $\mathcal{S}(330)$ state is nothing but the $k = 4$ parafermion state introduced by Read and Rezayi.¹⁴ The edge excitations of the state are described by a charge density mode ρ_c and a $k = 4$ parafermion conformal field theory.

V. SUMMARY

In this paper, we discuss another route to obtain non-Abelian FQH states through double-layer FQH states. In particular, we propose a possibility to start with the (330) double layer state, and then increase the interlayer

tunneling strength. We argue that such a process may change the (330) state to the $\mathcal{S}(330)$ state. Through the ideal wave function of the $\mathcal{S}(330)$ state, we find that the $\mathcal{S}(330)$ state is actually the Z_4 parafermion states proposed by Read and Rezayi¹⁴(RR). We demonstrate the equivalence of the two states in appendix B.

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Appendix A: The edge excitations of the Z_4 Read-Rezayi parafermion state

The edge excitations for the $(nn0)$ state can be described through $\rho_I(x)$, $I = 1, 2$, which are the 1D electron densities on the edge. ρ_1 is the electron density for the first layer and ρ_2 for the second layer. The Hilbert space and the dynamics of the edge excitations are described by the following current algebra (in the k -space)

$$[\rho_{Ik}, \rho_{Jk'}] = \frac{1}{n} \frac{1}{2\pi} k \delta_{k+k'} \delta_{IJ} \quad (A1)$$

and the Hamiltonian

$$H = \sum_{k>0} V_{IJ} \rho_{I,-k} \rho_{J,k}. \quad (A2)$$

eqn. (A1) and eqn. (A2) provide a complete description of the edge excitations. Note that eqn. (A1) and eqn. (A2) just describe a collection of harmonic oscillators labeled by $k > 0$ and $I = 1, 2$, with the lowering operator $a_{Ik} \propto \rho_{Ik}$ and raising operator $a_{Ik}^\dagger \propto \rho_{I,-k}$.

The electron operators on the edge are given by

$$\psi_{eI} = e^{i n \phi_I(x)}$$

where $\frac{1}{2\pi} \partial_x \phi(x) = \rho_I(x)$. ϕ_{e1} is for the first layer and ϕ_{e2} the second layer. The electron operators have the following correlation

$$\langle \psi_{eI}(t, 0) \psi_{eI}^\dagger(0, 0) \rangle \sim \frac{1}{t^{g_e}}, \quad g_e = n.$$

After we obtain the edge theory for the $(nn0)$ state and identify the electron operator, we are ready to do the projection into the even states and to obtain the edge theory for the $\mathcal{S}(nn0)$ state. To do so, we first identify a new electron operator ψ_e and then use the new electron operator and only the new electron operator to create the edge excitations of the single-layer $\mathcal{S}(nn0)$ state.

Since the $\mathcal{S}(nn0)$ state only contains electrons in the even state, so the new electron operator is

$$\psi_e(x) = \psi_{e1}(x) + \psi_{e2}(x).$$

The other combination $\psi_{e1}(x) - \psi_{e2}(x)$ does not generate gapless edge excitations and is dropped. If we use ψ_{e1} and ψ_{e2} to generate gapless edge excitations, we will generate all the edge excitations of the $(nn0)$ state. However, to

obtain the gapless edge excitations for the $\mathcal{S}(nn0)$ state, we can only use $\psi_e = \psi_{e1} + \psi_{e2}$. So the $\mathcal{S}(nn0)$ state will have fewer gapless edge excitations.

What kind of edge excitations does ψ_e generate? To answer such a question, we introduce the total electron density ρ_c and the relative electron density ρ_r of the two layers:

$$\rho_c = \rho_1 + \rho_2, \quad \rho_r = \rho_1 - \rho_2.$$

ρ_c and ρ_r satisfy the following current algebra

$$\begin{aligned} [\rho_{ck}, \rho_{ck'}] &= \frac{2}{n} \frac{1}{2\pi} k \delta_{k+k'} \\ [\rho_{rk}, \rho_{rk'}] &= \frac{2}{n} \frac{1}{2\pi} k \delta_{k+k'} \end{aligned}$$

In terms of ϕ_c and ϕ_r , defined through $\frac{1}{2\pi} \partial_x \phi_c = \rho_c$ and $\frac{1}{2\pi} \partial_x \phi_r = \rho_r$, the new electron operator has the form

$$\psi_e = e^{in\phi_c/2} \cos(n\phi_r/2).$$

From the relation between the FQH wave function and CFT,^{1,7} the correlation of the above electron operator reproduces the $\mathcal{S}(nn0)$ wave function

$$\Phi_{\mathcal{S}(nn0)}(\{z_i\}) = \langle V(z_\infty) \prod_i \psi_e(z_i) \rangle. \quad (\text{A3})$$

First, let us consider the case for $n = 3$. We have shown (see appendix B) that $\Phi_{\mathcal{S}(330)}(\{z_i\})$ is the Z_4 Read-Rezayi parafermion state. This means that $\Phi_{\mathcal{S}(330)}(\{z_i\})$ can be expressed as a correlation function in the Z_4 parafermion CFT:

$$\Phi_{\mathcal{S}(330)}(\{z_i\}) = \langle V(z_\infty) \prod_i \psi_1(z_i) e^{i3\phi_c(z_i)/2} \rangle, \quad (\text{A4})$$

where ψ_1 is the simple current operator that generates the Z_4 parafermion CFT. We see that

$$\psi_e(z) = \psi_1(z) e^{i3\phi_c(z)/2} \quad (\text{A5})$$

and we can identify $\cos(3\phi_r/2)$ with ψ_1 . In fact, $\cos(3\phi_r/2)$ has a scaling dimension $3/4$ which matches that of ψ_1 .

The operator product expansion of the $\psi_e(z) = \psi_1(z) e^{in\phi_c(z)/2}$ will generate the operator ρ_c , $\psi_1 \psi_1^\dagger \sim \psi_1(\psi_1)^3$ etc. So the edge excitations (with a fixed total electron number) form a Hilbert space \mathcal{H} which is the direct product of two Hilbert spaces: $\mathcal{H} = \mathcal{H}_{U(1)} \otimes \mathcal{H}_{Z_4}$. $\mathcal{H}_{U(1)}$ is generated by ρ_c and \mathcal{H}_{Z_4} is generated by $\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)$.

What is the spectrum of the edge excitations generated by ρ_c and $\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)$? Let M_0 be the total angular momentum of the ground state wave function $\Phi_{\mathcal{S}(330)}$. The number of the low-energy edge states at angular momentum $M_0 + l$ is given by D_l . We can introduce a function

$$\text{ch}(\xi) = \sum_{l=0}^{\infty} D_l \xi^l$$

to describe the edge spectrum D_l . The function is called the character of the edge excitations.

If we only use ρ_c to generate edge excitations, the character will be

$$\text{ch}_c(\xi) = \frac{1}{\prod_{l=1}^{\infty} (1 - \xi^l)}.$$

If we only use $\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)$ to generate edge excitations, the character will be that of Z_4 parafermion CFT.²¹ The character (in the vacuum sector) for the Z_k parafermion CFT is given by

$$\begin{aligned} \text{ch}_{Z_k}(\xi) &= [\text{ch}_c(\xi)]^2 \times \\ &\sum_{r,s=0}^{\infty} (-1)^{r+s} \xi^{r(r+1)+s(s+1)+rs(k+1)} \left[1 - \xi^{(k+1)(1+r+s)} \right]. \end{aligned} \quad (\text{A6})$$

If we apply both ρ_c and $\psi_1(z_1)\psi_1(z_2)\psi_1(z_3)\psi_1(z_4)$ to generate edge excitations, the character will be

$$\text{ch}(\xi) = \text{ch}_c(\xi) \text{ch}_{Z_4}(\xi). \quad (\text{A7})$$

The character $\text{ch}(\xi)$ describes the edge spectrum of the $\mathcal{S}(330)$ or Z_4 Read-Rezayi parafermion state.

We find that the edge spectrum for the $\mathcal{S}(330)$ state is given by

$$D_l : 1, 1, 3, 6, 13, 23, 44, 75, 131, 215, 354, 561 \dots$$

The electron operators have the following correlation

$$\langle \psi_e(t, 0) \psi_e^\dagger(0, 0) \rangle \sim \frac{1}{t^{g_e}}, \quad g_e = n.$$

Thus $g_e = 3$ for the $\mathcal{S}(330)$ state.

A quasiparticle operator ψ_q must satisfy the following condition: in the operator product between ψ_q and ψ_e

$$\psi_e(x) \psi_q(0) = \sum_i \frac{1}{x^{\alpha_i}} O_i(0),$$

where the exponents α_i must all be integers. In this case, we say that the quasiparticle operator ψ_q is mutually local with respect to the electron operator ψ_e .²⁰ One of the quasiparticles in the $\mathcal{S}(nn0)$ state is created by

$$\psi_q^{(\frac{1}{2}, \frac{1}{2})} = e^{i\phi_c/2} \cos(\phi_r/2)$$

which carries charge e/n . The quasiparticle operator has the following correlation

$$\langle \psi_q^{(\frac{1}{2}, \frac{1}{2})}(t, 0) \psi_q^{(\frac{1}{2}, \frac{1}{2})\dagger}(0, 0) \rangle \sim \frac{1}{t^{g_q}}, \quad g_q^{(\frac{1}{2}, \frac{1}{2})} = 1/n.$$

For the Z_4 state, we find $g_q^{(\frac{1}{2}, \frac{1}{2})} = 1/3$ and $Q_q^{(\frac{1}{2}, \frac{1}{2})} = e/3$.

Similarly $\psi_q^{(k,l)}$ defined by

$$\psi_q^{(k,l)} = e^{ik\phi_c} \cos[l\phi_r], \quad 2k = \text{integer}, \quad 2l = \text{integer},$$

are also valid quasiparticle operators, which carry charge $Q^{(k,l)} = 2ke/n$. The exponent for such a quasiparticle is given by $g_q^{(k,l)} = \frac{1}{2n}(k^2 + l^2)$.

However, the quasiparticle $\psi_q^{(\frac{1}{2}, \frac{1}{2})}$ is not the one that carries minimal charge. The minimally charged quasiparticle has a charge $Q_q = e/6$ and a scaling dimension $h_q = \frac{1}{16} + \frac{1}{48} = \frac{1}{4}$. So the exponent for such a quasiparticle is $g_q = 2h_q = 1/2$.

Now, let us consider the case for $n = 2$. We can fermionize the ρ_r sector. Introducing a complex fermion field $\psi(x)$, the states generated by $\rho_r(x)$ can be equally generated by $\psi^\dagger(x)\psi(x)$. In the fermion description $e^{i2\phi_r(x)} \sim \psi(x)$. Thus $\cos[2\phi_r(x)] \sim \psi(x) + \psi^\dagger(x) \equiv \lambda(x)$ where $\lambda(x)$ is a Majorana fermion field which satisfies

$$\lambda(x)\lambda(0) = \frac{1}{x} + x[\partial\lambda(0)]\lambda(0) + \dots$$

Using $\psi_e = e^{i2\phi_c}\lambda$, we find that

$$\begin{aligned} \psi_e(x)\psi_e^\dagger(0) &= x^{-2} + i4\pi\rho_c(0)x^{-1} \\ &\quad - 8\pi^2\rho_c^2(0) - \lambda(0)\partial\lambda(0) + \dots \end{aligned}$$

Thus the edge excitations for the $n = 2$ case are generated by ρ_c and $\lambda\partial\lambda$. These edge excitations are described by a density mode ρ_c and a Majorana fermion λ .¹²

If we only use ρ_c to generate edge excitations, the character will be

$$\text{ch}_c(\xi) = \frac{1}{\prod_{l=1}^{\infty}(1 - \xi^l)}.$$

If we only use $\lambda\partial\lambda$ to generate edge excitations, the character will be $\text{ch}_{Z_2}(\xi)$. If we apply both ρ_c and $\lambda\partial\lambda$ to generate edge excitations, the character will be

$$\text{ch}(\xi) = \text{ch}_c(\xi)\text{ch}_{Z_2}(\xi). \quad (\text{A8})$$

We find that¹²

$$D_l : 1, 1, 3, 5, 10, 16, 28, 43, 70, 105, 161, 236, \dots$$

Appendix B: Relation to the Z_4 RR state

To show the equivalence the of $\mathcal{S}(330)$ and Z_4 RR state we will compare the boson version of the two states by dividing out a Jastrow factor $\prod_{i < j} (z_i - z_j)$ from the former. We will invoke the well known property of the z_k RR states: as a function of their complex coordinates the wave functions vanish quadratically when $k + 1$ particles approach. We will also assume that $\mathcal{S}(330)$ does not vanish identically which, as shown below, is not the case for $N/2$ odd. This is to be expected since the RR state for $k = 4$ does not exist for odd $N/2$. Starting with $\mathcal{S}(330)$ we write it as

$$\begin{aligned} &\Psi_{\mathcal{S}(330)}(z_1, z_2, \dots, z_N) \\ &= \sum_Q (-)^Q \prod_{Q_i < Q_j} (z_{Q_i} - z_{Q_j})^3 \prod_{Q_m < Q_n} (z_{Q_m} - z_{Q_n})^3 \end{aligned} \quad (\text{B1})$$

where Q is a permutation of N objects, and i and $j \neq m$ or n . We have omitted the exponential factors. We have used the antisymmetry of the Laughlin terms to restrict the number of permutations to a minimum. We will let 5 particles approach each other while keeping the other $N - 5$ well separated from this group and each other. We relabel the coordinates $\omega_1, \omega_2, \dots, \omega_5$. Then in eqn. (B1) there are 5 types of terms depending on how these 5 particles are distributed in the two Laughlin factors: $[5, 0]$ (and $[0, 5]$), $[4, 1]$, and $[3, 2]$. It can be seen that the total number of such terms respectively are:

$$2 \binom{N-5}{N/2}, 2 \binom{N-5}{N/2-1} \binom{5}{4}, \text{ and } 2 \binom{N-5}{N/2-2} \binom{5}{3}.$$

It is straightforward to show that the sum of the above is $\binom{N}{N/2}$, which is the number of terms in eqn. (B1).

First consider the $[5, 0]$ terms. After dividing out the Jastrow factor it is clear that these terms vanish if any two of the five particles coordinates are equal. For the $[4, 1]$ terms, a little algebra shows that they vanish if 4 of the coordinates are set to be equal. We are left with $[3, 2]$ terms which contain factors such as

$$(\omega_1 - \omega_2)^3 (\omega_1 - \omega_3)^3 (\omega_2 - \omega_3)^3 (\omega_4 - \omega_5)^3. \quad (\text{B2})$$

In the rest of the wavefunction we set the ω_i 's to their common values ω without loss of generality. We can then collect the terms in Q that permute the five particles amongst themselves ($\binom{5}{3} = 10$ terms). Dividing out the Jastrow factor, we obtain

$$-3 \sum_{i < j}^5 (\omega_i - \omega_j)^2. \quad (\text{B3})$$

We have included an equal expression from the $[2, 3]$ distribution of the ω 's. To see that $[3, 2]$ and $[2, 3]$ are identical for even $N/2$ consider the two permutations that interchange the Laughlin terms:

$$\begin{aligned} Q &= I : (1, 2, 3, \dots, N/2; N/2 + 1, N/2 + 2, \dots, N), \\ Q' &: (N/2 + 1, N/2 + 2, \dots, N : 1, 2, \dots, N/2). \end{aligned}$$

Clearly, these permutations produce identical terms but, depending on whether $N/2$ is even/odd, they have the same/opposite parities. Since every permutation can be paired in this way, it can be seen that exchanging the Laughlin factors preserves the overall sign for even $N/2$, leading to the equality of $[3, 2]$ and $[2, 3]$. On the other hand, for odd $N/2$ with opposite signs, eqn. (B1) identically vanishes.

So far we have established that the two functions are equivalent but this still leaves out an overall multiplicative factor. We have empirically determined that the factor is

$$\frac{\Psi_{RR}(z_1, z_2, \dots, z_N)}{\Psi_{\mathcal{S}(330)}(z_1, z_2, \dots, z_N)} = (-1)^{\frac{N/4(N/4+1)}{2}} 4^{N/4}, \quad (\text{B4})$$

where we have used the following form for the Z_4 wave-function:

$$\begin{aligned} & \Psi_{RR}(z_1, z_2, \dots, z_N) \\ &= \sum_Q \prod_{Q_i < Q_j} (z_{Q_i} - z_{Q_j})^2 \prod_{Q_k < Q_l} (z_{Q_k} - z_{Q_l})^2 \times \\ & \quad \prod_{Q_m < Q_n} (z_{Q_m} - z_{Q_n})^2 \prod_{Q_r < Q_p} (z_{Q_r} - z_{Q_p})^2 \quad (\text{B5}) \end{aligned}$$

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